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Research on the Dynamic Problems of 3D Cross Coupling Quantum Harmonic Oscillator by Virtue of Intermediate Representation $|x\rangle_{\lambda,\nu}$

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Abstract The intermediate representation (namely intermediate coordinate-momentum representation) $|x\rangle_{\lambda,\nu}$ are introduced and employed to research the expression of the operator $\tau\hat{p}+\sigma\hat{x}$ in intermediate representation $|x\rangle_{\lambda,\nu}$. The systematic Hamilton operator $\hat{H}$ of 3D cross coupling quantum harmonic oscillator was diagonalized by virtue of quadratic form theory. The quantity of λ, ν, $\tau$and σ were figured out. The dynamic problems of 3D cross coupling quantum harmonic oscillator are researched by virtue of intermediate representation. The energy eigen-value and eigenwave function of 3D cross coupling quantum harmonic oscillator were obtained in intermediate representation. The importance of intermediate representation was discussed. The results show that the Radon transformation of Wigner operator is just the projectional operator $|x\rangle_{\lambda,\nu\lambda,\nu}\langle x|$, and the Radon transformation of Wigner function is just a margin distribution.

Keywords Intermediate representation · Quadratic form

1 Introduction

Much interest has been paid to the dynamic problems of coupling harmonic oscillator in quantum mechanics, quantum optics, molecule spectrum and mesoscopic coupling circuit quantization. Zeng [1] studied on coordinate coupling double harmonic oscillators by perturbation theory and coordinate transformation. Ji and Lei [2] studied on diagonalization of Hamiltonian for three harmonically coupling non-identical oscillators, and [3] gave the exact solution for non-identical n modes coupling harmonic oscillators. These researches are about coordinate coupling, and not related with momentum coupling. In [4–12] we only

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investigated 2D double coupling, respectively. In fact, lots of practical physics problems are equivalent to system of coordinate and momentum cross coupling 3D harmonic oscillators. The authors of [13–16] had studied representation widely. But the dynamics of cross coupling 3D harmonic oscillators in intermediate representation has not been published yet. In this paper, the intermediate representation $|x\rangle_{\lambda,\nu}$ is employed to study dynamics of cross coupling 3D harmonic oscillators. The systematic Hamilton operator $\hat{H}$ was diagonalized by virtue of quadratic form theory. The quantity of λ, ν, τ and σ were figured out. The energy eigen-value and eigenwave function of 3D cross coupling quantum harmonic oscillator were obtained in intermediate representation. The results show that the Radon transformation of Wigner operator is just the projectional operator $|x\rangle_{\lambda,\nu\lambda,\nu}\langle x|$, and the Radon transformation of Wigner function is just a margin distribution.

2 Expression of the $\hat{P} = (\tau\hat{p} + \sigma\hat{x})$ in Intermediate Representation $|x\rangle_{\lambda,\nu}$

The $|x\rangle_{\lambda,\nu}$ denotes eigenvector of operator $\hat{X} = (\lambda\hat{x} + \nu\hat{p})$,

$$(\lambda\hat{x} + \nu\hat{p})|x\rangle_{\lambda,\nu} = x|x\rangle_{\lambda,\nu}. \tag{1}$$

In Fock space $|x\rangle_{\lambda,\nu}$ can be expressed as [17]

$$|x\rangle_{\lambda,\nu} = \left\{\frac{\pi\hbar}{m\omega}[\lambda^2 + (m\omega\nu)^2]\right\}^{-1/4} \exp\left[-\frac{m\omega}{2\hbar}\frac{x^2}{\lambda^2 + (m\omega\nu)^2} + \sqrt{\frac{2m\omega}{\hbar}}\frac{x}{\lambda - \mathrm{i}m\omega\nu}a^\dagger - \frac{\lambda + \mathrm{i}m\omega\nu}{2(\lambda - \mathrm{i}m\omega\nu)}a^{\dagger 2}\right]|0\rangle, \tag{2}$$

which is an orthogonal and self-contained intermediate representation.

The operator $(\lambda\hat{x} + \nu\hat{p})$ and $\hat{P} = (\tau\hat{p} + \sigma\hat{x})$ satisfy the following commutation relation [17]

$$[(\lambda\hat{x} + \nu\hat{p}), (\tau\hat{p} + \sigma\hat{x})] = (\lambda\tau - \nu\sigma)[\hat{x}, \hat{p}] = (\lambda\tau - \nu\sigma)\mathrm{i}\hbar. \tag{3}$$

If $(\lambda\tau - \nu\sigma) = 1$, the operator $(\lambda\hat{x} + \nu\hat{p})$ and $(\tau\hat{p} + \sigma\hat{x})$ are a couple conjugate operators

$${}_{\lambda,\nu}\langle x|(\lambda\hat{x} + \nu\hat{p}) = x\,{}_{\lambda,\nu}\langle x|. \tag{4}$$

Using the following relations [7]

$${}_{\lambda,\nu}\langle x \mid x'\rangle = (2\pi\hbar\nu)^{-1/2}\exp\left(-\mathrm{i}\frac{xx'}{\hbar\nu} + \frac{\mathrm{i}}{2}\frac{\lambda}{\hbar\nu}x'^2\right), \tag{5}$$

we can obtain

$$\begin{aligned} {}_{\lambda,\nu}\langle x|(\tau\hat{p} + \sigma\hat{x})|x'\rangle &= {}_{\lambda,\nu}\langle x|\left(\mathrm{i}\tau\hbar\frac{\partial}{\partial x'} + \sigma x'\right)|x'\rangle = \left(\mathrm{i}\tau\hbar\frac{\partial}{\partial x'} + \sigma x'\right){}_{\lambda,\nu}\langle x|x'\rangle \\ &= \left(\frac{\tau}{\nu}x - \frac{1}{\nu}x'\right){}_{\lambda,\nu}\langle x|x'\rangle. \end{aligned} \tag{6}$$

With the aid of

$$-\mathrm{i}\hbar\frac{\partial}{\partial x}{}_{\lambda,\nu}\langle x|x'\rangle = -\frac{1}{\nu}x'{}_{\lambda,\nu}\langle x|x'\rangle \tag{7}$$

and comparing (6) and (7), we obtain

$$_{\lambda,\nu}\langle x|\hat{P} = {}_{\lambda,\nu}\langle x|(\tau\hat{p}+\sigma\hat{x}) = \left(-\mathrm{i}\hbar\frac{\partial}{\partial x}+\frac{\tau}{\nu}x\right){}_{\lambda,\nu}\langle x|. \tag{8}$$

According to the virtue of Unitary transformation,

$$|x\rangle_{\lambda,\nu} \to |q\rangle_{\lambda,\nu} = e^{\mathrm{i}\frac{\tau}{2\nu\hbar}x^2}|x\rangle_{\lambda,\nu}. \tag{9}$$

Obviously, the expression of operator $(\lambda\hat{x}+\nu\hat{p})$ is fixedness in intermediate representation

$$\hat{X} = (\lambda\hat{x}+\nu\hat{p}) = x \to \hat{Q} = e^{\mathrm{i}\frac{\tau}{2\nu\hbar}x^2}\hat{X}e^{-\mathrm{i}\frac{\tau}{2\nu\hbar}x^2} = e^{\mathrm{i}\frac{\tau}{2\nu\hbar}x^2}xe^{-\mathrm{i}\frac{\tau}{2\nu\hbar}x^2} = x. \tag{10}$$

But the change of operator $(\tau\hat{p}+\sigma\hat{x})$ is

$$\hat{P} = (\tau\hat{p}+\sigma\hat{x}) = -\mathrm{i}\hbar\frac{\partial}{\partial x}+\frac{\tau}{\nu}x \to \hat{\Phi} = e^{\mathrm{i}\frac{\tau}{2\nu\hbar}x^2}\hat{P}e^{-\mathrm{i}\frac{\tau}{2\nu\hbar}x^2} = -\mathrm{i}\hbar\frac{\partial}{\partial x}. \tag{11}$$

3 The Energy Eigenvalue and Characteristic Function of 3D Cross Coupling Quantum Harmonic Oscillator

The systematic Hamilton operator of 3D cross coupling quantum harmonic oscillator is

$$\begin{aligned}\hat{H} = {}& \frac{\hat{p}_1^2}{2m_1}+\frac{\hat{p}_2^2}{2m_2}+\frac{\hat{p}_3^2}{2m_3}+\frac{1}{2}m_1\omega_1^2\hat{x}_1^2+\frac{1}{2}m_2\omega_2^2\hat{x}_2^2+\frac{1}{2}m_3\omega_3^2\hat{x}_3^2 \\ &+\gamma_1(\hat{x}_1\hat{p}_1+\hat{p}_1\hat{x}_1)+\gamma_2(\hat{x}_2\hat{p}_2+\hat{p}_2\hat{x}_2)+\gamma_3(\hat{x}_3\hat{p}_3+\hat{p}_3\hat{x}_3),\end{aligned} \tag{12}$$

where γ_1, γ_2 and γ_3 denote coupling intensity of each directional coordinate and momentum, respectively. Equation (12) is a quadratic form, which can be written as the following matrix form

$$\hat{H} = \chi^T A\chi, \tag{13}$$

where

$$\chi = \begin{pmatrix}\hat{p}_1\\ \hat{p}_2\\ \hat{p}_3\\ m_1\omega_1\hat{x}_1\\ m_2\omega_2\hat{x}_2\\ m_3\omega_3\hat{x}_3\end{pmatrix}, \qquad A = \begin{pmatrix}\frac{1}{2m_1} & 0 & 0 & \frac{\gamma_1}{m_1\omega_1} & 0 & 0\\ 0 & \frac{1}{2m_2} & 0 & 0 & \frac{\gamma_2}{m_2\omega_2} & 0\\ 0 & 0 & \frac{1}{2m_3} & 0 & 0 & \frac{\gamma_3}{m_3\omega_3}\\ \frac{\gamma_1}{m_1\omega_1} & 0 & 0 & \frac{1}{2m_1} & 0 & 0\\ 0 & \frac{\gamma_2}{m_2\omega_2} & 0 & 0 & \frac{1}{2m_2} & 0\\ 0 & 0 & \frac{\gamma_3}{m_3\omega_3} & 0 & 0 & \frac{1}{2m_3}\end{pmatrix}.$$

Matrix A is the matrix of quadratic form, and its characteristic equation is

$$\begin{pmatrix}\frac{1}{2m_1}-\Lambda & 0 & 0 & \frac{\gamma_1}{m_1\omega_1} & 0 & 0\\ 0 & \frac{1}{2m_2}-\Lambda & 0 & 0 & \frac{\gamma_2}{m_2\omega_2} & 0\\ 0 & 0 & \frac{1}{2m_3}-\Lambda & 0 & 0 & \frac{\gamma_3}{m_3\omega_3}\\ \frac{\gamma_1}{m\omega_1} & 0 & 0 & \frac{1}{2m_1}-\Lambda & 0 & 0\\ 0 & \frac{\gamma_2}{m_2\omega_2} & 0 & 0 & \frac{1}{2m_2}-\Lambda & 0\\ 0 & 0 & \frac{\gamma_3}{m_3\omega_3} & 0 & 0 & \frac{1}{2m_3}-\Lambda\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\\ c_4\\ c_5\\ c_6\end{pmatrix} = 0. \tag{14}$$

The secular equation can be written as

$$\begin{vmatrix} \frac{1}{2m_1}-\Lambda & 0 & 0 & \frac{\gamma_1}{m_1\omega_1} & 0 & 0 \\ 0 & \frac{1}{2m_2}-\Lambda & 0 & 0 & \frac{\gamma_2}{m_2\omega_2} & 0 \\ 0 & 0 & \frac{1}{2m_3}-\Lambda & 0 & 0 & \frac{\gamma_3}{m_3\omega_3} \\ \frac{\gamma_1}{m_1\omega_1} & 0 & 0 & \frac{1}{2m_1}-\Lambda & 0 & 0 \\ 0 & \frac{\gamma_2}{m_2\omega_2} & 0 & 0 & \frac{1}{2m_2}-\Lambda & 0 \\ 0 & 0 & \frac{\gamma_3}{m_3\omega_3} & 0 & 0 & \frac{1}{2m_3}-\Lambda \end{vmatrix} = 0. \tag{15}$$

From (14) and (15), we can derive six latent roots and corresponding orthogonal and normalized eigenvectors.

$$\Lambda_1 = \frac{1}{2m_1} - \frac{\gamma_1}{m_1\omega_1}, \qquad \vec{\xi}_1 = \frac{1}{\sqrt{2}}(1 \quad 0 \quad 0 \quad -1 \quad 0 \quad 0)^T, \tag{16}$$

$$\Lambda_2 = \frac{1}{2m_2} - \frac{\gamma_2}{m_2\omega_2}, \qquad \vec{\xi}_2 = \frac{1}{\sqrt{2}}(0 \quad 1 \quad 0 \quad 0 \quad -1 \quad 0)^T, \tag{17}$$

$$\Lambda_3 = \frac{1}{2m_3} - \frac{\gamma_3}{m_3\omega_3}, \qquad \vec{\xi}_3 = \frac{1}{\sqrt{2}}(0 \quad 0 \quad 1 \quad 0 \quad 0 \quad -1)^T, \tag{18}$$

$$\Lambda_4 = \frac{1}{2m_1} + \frac{\gamma_1}{m_1\omega_1}, \qquad \vec{\xi}_4 = \frac{1}{\sqrt{2}}(1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0)^T, \tag{19}$$

$$\Lambda_5 = \frac{1}{2m_2} + \frac{\gamma_2}{m_2\omega_2}, \qquad \vec{\xi}_5 = \frac{1}{\sqrt{2}}(0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0)^T, \tag{20}$$

$$\Lambda_6 = \frac{1}{2m_3} + \frac{\gamma_3}{m_3\omega_3}, \qquad \vec{\xi}_6 = \frac{1}{\sqrt{2}}(0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1)^T. \tag{21}$$

By constructing unitary matrix $S = (\vec{\xi}_1 \quad \vec{\xi}_2 \quad \vec{\xi}_3 \quad \vec{\xi}_4 \quad \vec{\xi}_5 \quad \vec{\xi}_6)$ and performing transformations $\chi = S\chi'$, namely

$$\begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \\ m_1\omega_1\hat{x}_1 \\ m_2\omega_2\hat{x}_2 \\ m_3\omega_3\hat{x}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \hat{P}_1 \\ \hat{P}_2 \\ \hat{P}_3 \\ m_1\omega_1\hat{X}_1 \\ m_2\omega_2\hat{X}_2 \\ m_3\omega_3\hat{X}_3 \end{pmatrix}, \tag{22}$$

(13) will become

$$\hat{H} = \chi^T A \chi = \chi'^T S^T A S \chi' = \chi'^T \Lambda \chi' \tag{23}$$

where

$$\chi' = \begin{pmatrix} \hat{P}_1 \\ \hat{P}_2 \\ \hat{P}_3 \\ m_1\omega_1\hat{X}_1 \\ m_2\omega_2\hat{X}_2 \\ m_3\omega_3\hat{X}_3 \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} \frac{1}{2m_1}-\frac{\gamma_1}{m_1\omega_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2m_2}-\frac{\gamma_2}{m_2\omega_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2m_3}-\frac{\gamma_3}{m_3\omega_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2m_1}+\frac{\gamma_1}{m_1\omega_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2m_2}+\frac{\gamma_2}{m_2\omega_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2m_3}+\frac{\gamma_3}{m_3\omega_3} \end{pmatrix}.$$

According to (23), we obtain

$$\hat{H} = \sum_{i=1}^{3}\left\{\frac{\hat{P}_i^2}{2\mu_i} + \frac{1}{2}\mu_i\Omega_i^2\hat{X}_i^2\right\} \tag{24}$$

where

$$\mu_1 = \frac{m_1\omega_1}{\omega_1 - 2\gamma_1}, \qquad \Omega_1 = \sqrt{\omega_1^2 - 4\gamma_1^2}, \tag{25}$$

$$\mu_2 = \frac{m_2\omega_2}{\omega_2 - 2\gamma_2}, \qquad \Omega_2 = \sqrt{\omega_2^2 - 4\gamma_2^2}, \tag{26}$$

$$\mu_3 = \frac{m_3\omega_3}{\omega_3 - 2\gamma_3}, \qquad \Omega_3 = \sqrt{\omega_3^2 - 4\gamma_3^2}. \tag{27}$$

The inverse transformations of (22) are

$$\begin{cases} \hat{P}_1 = \frac{1}{\sqrt{2}}(\hat{p}_1 - m_1\omega_1\hat{x}_1), \\ \hat{P}_2 = \frac{1}{\sqrt{2}}(\hat{p}_2 - m_2\omega_2\hat{x}_2), \\ \hat{P}_3 = \frac{1}{\sqrt{2}}(\hat{p}_3 - m_3\omega_3\hat{x}_3) \end{cases} \quad \text{and} \quad \begin{cases} \hat{X}_1 = \frac{1}{\sqrt{2}}\left(\frac{\hat{p}_1}{m_1\omega_1} + \hat{x}_1\right), \\ \hat{X}_2 = \frac{1}{\sqrt{2}}\left(\frac{\hat{p}_2}{m_2\omega_2} + \hat{x}_2\right), \\ \hat{X}_3 = \frac{1}{\sqrt{2}}\left(\frac{\hat{p}_3}{m_3\omega_3} + \hat{x}_3\right). \end{cases} \tag{28}$$

Hereto, we have diagonalized Hamilton operator $\hat{H}$ by virtue of quadratic form theory, and figured out the quantity of λ, ν, τ and σ

$$\lambda_i = \frac{1}{\sqrt{2}}, \quad \nu_i = \frac{1}{\sqrt{2}m_i\omega_i}, \quad \tau_i = \frac{1}{\sqrt{2}}, \quad \sigma_i = -\frac{m_i\omega_i}{\sqrt{2}} \quad (i = 1, 2, 3). \tag{29}$$

By Unitary transformation of (24), we derive

$$\hat{H}' = e^{\sum_{i=1}^{3}\mathrm{i}\frac{\tau_i}{2\nu_i\hbar}x_i^2}\,\hat{H}\,e^{\sum_{i=1}^{3}(-\mathrm{i}\frac{\tau_i}{2\nu_i\hbar}x_i^2)} = \sum_{i=1}^{3}\left\{\frac{\hat{\Phi}_i^2}{2\mu_i} + \frac{1}{2}\mu_i\Omega_i^2\hat{Q}_i^2\right\}. \tag{30}$$

By operating $\hat{H}'$ on intermediate representation $\prod_{i=1}^{3}|q_i\rangle_{\lambda_i,\nu_i}$

$$\prod_{i=1}^{3}{}_{\lambda_i,\nu_i}\langle q_i|\hat{H} = \sum_{i=1}^{3}\left\{-\frac{\hbar^2}{2\mu_i}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\mu_i\Omega_i^2x_i^2\right\}\prod_{i=1}^{3}{}_{\lambda_i,\nu_i}\langle q_i| \tag{31}$$

we can immediately know its eigenvalue spectrum and eigenwave function

$$\begin{cases} E_{n_1n_2n_3} = \sum_{i=1}^{3}\left(n_i + \frac{1}{2}\right)\hbar\Omega_i, \\ \prod_{i=1}^{3} {}_{\lambda_i,\nu_i}\langle q_i|\Psi_{n_1n_2n_3}\rangle = \prod_{i=1}^{3} e^{-i\frac{m_i\omega_i}{2\hbar}x_i^2}{}_{\lambda_i,\nu_i}\langle x_i|\Psi_{n_1n_2n_3}\rangle \\ \qquad\qquad = \prod_{i=1}^{3}\left\{N_{n_i}e^{-\frac{\alpha_i^2}{2}x_i^2}H_{n_i}(\alpha_i x_i)\right\}, \end{cases} \tag{32}$$

where $\alpha_i = \sqrt{\frac{\mu_i\Omega_i}{\hbar}}$. In coordinate representation eigenwave function is

$$\prod_{i=1}^{3}\langle x_i|\Psi_{n_1n_2n_3}\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_1'dx_2'dx_3' \prod_{i=1}^{3}\langle x_i|x_i'\rangle_{\lambda_i,\nu_i}\prod_{k=1}^{3}{}_{\lambda_k,\nu_k}\langle x_k'|\Psi_{n_1n_2n_3}\rangle. \tag{33}$$

Substituting (29) into (5) and (34) into (33), respectively, we have

$$\langle x_i|x_i'\rangle_{\lambda_i,\nu_i} = \left(\frac{m_i\omega_i}{\sqrt{2}\pi\hbar}\right)^{\frac{1}{2}}\exp\left(\mathrm{i}\sqrt{2}\frac{m_i\omega_i x}{\hbar}x' - \frac{\mathrm{i}}{2}\frac{m_i\omega_i}{\hbar}x_i^2\right) \quad (i = 1, 2, 3), \tag{34}$$

$$\prod_{i=1}^{3}\langle x_i|\Psi_{n_1n_2n_3}\rangle = \left\{\prod_{i=1}^{3}\left[\sqrt{\frac{m_i\omega_i}{\sqrt{2}\pi\hbar}}N_{n_i}\exp\left(-\frac{\mathrm{i}}{2}\frac{m_i\omega_i}{\hbar}x_i^2\right)\right]\right\}\prod_{i=1}^{3}F_{n_i}(x_i), \tag{35}$$

where

$$\begin{aligned} F_{n_i}(x_i) &= \int_{-\infty}^{\infty} dx_i' \exp\left[-\frac{1}{2}\left(\alpha_i^2 - \mathrm{i}\frac{m_i\omega_i}{\hbar}\right)x_i'^2 + \mathrm{i}\frac{\sqrt{2}m_i\omega_i}{\hbar}x_i x_i'\right]H_n(\alpha_i x_i') \\ &= \frac{1}{\alpha_i}\int_{-\infty}^{\infty} d\xi_i \exp\left[-\frac{\xi_i^2}{2}\left(1 - \mathrm{i}\frac{m_i\omega_i}{\hbar\alpha_i^2}\right) + \mathrm{i}\frac{\sqrt{2}m_i\omega_i}{\alpha_i}x_i\xi_i\right]H_{n_i}(\xi_i) \\ &= \sqrt{\frac{2\pi\hbar}{\hbar\alpha_i^2 - \mathrm{i}m_i\omega_i}}\exp\left(-\frac{m_i^2\omega_i^2}{\hbar^2\alpha_i^2 - \mathrm{i}\hbar m_i\omega_i}x_i^2\right)\frac{d^n}{dt^n} \\ &\quad \times \exp\left(\frac{\hbar\alpha_i^2 + \mathrm{i}m_i\omega_i}{\hbar\alpha_i^2 - \mathrm{i}m_i\omega_i}t^2 + \mathrm{i}\frac{2\sqrt{2}m_i\omega_i\alpha_i}{\hbar\alpha_i^2 - \mathrm{i}m_i\omega_i}x_i t\right)\Bigg|_{t=0} \end{aligned} \tag{36}$$

and we have used the formula $H_n(\xi) = \frac{d^n}{dt^n}\exp(2\xi t - t^2)|_{t=0}$.

The front three $F_{n_i}(x_i)$ have forms as follows

$$F_0(x_i) = \sqrt{\frac{2\pi\hbar}{\hbar\alpha_i^2 - \mathrm{i}m_i\omega_i}}\exp\left(-\frac{m_i^2\omega_i^2}{\hbar^2\alpha_i^2 - \mathrm{i}\hbar m_i\omega_i}x_i^2\right),$$

$$F_1(x_i) = \sqrt{\frac{2\pi\hbar}{\hbar\alpha_i^2 - \mathrm{i}m_i\omega_i}}\left(\mathrm{i}\frac{2\sqrt{2}m_i\omega_i\alpha_i}{\hbar\alpha_i^2 - \mathrm{i}m_i\omega_i}x_i\right)\exp\left(-\frac{m_i^2\omega_i^2}{\hbar^2\alpha_i^2 - \mathrm{i}\hbar m_i\omega_i}x_i^2\right),$$

$$F_2(x_i) = \sqrt{\frac{2\pi\hbar}{\hbar\alpha_i^2 - \mathrm{i}m_i\omega_i}}\left[\frac{2(\hbar\alpha_i^2 + \mathrm{i}m_i\omega_i)}{\hbar\alpha_i^2 - \mathrm{i}m_i\omega_i} - \left(\frac{2\sqrt{2}m_i\omega_i\alpha_i}{\hbar\alpha_i^2 - \mathrm{i}m_i\omega_i}\right)^2 x_i^2\right]$$

$$\times \exp\left(-\frac{m_i^2\omega_i^2}{\hbar^2\alpha_i^2 - \mathrm{i}\hbar m_i\omega_i}x_i^2\right),$$
$$\ldots$$

When $\gamma_1 = \gamma_2 = \gamma_3 = 0$, we have

$$F_0(x_i) = \sqrt{\frac{2\pi\hbar}{m_i\omega_i(1-\mathrm{i})}}\exp\left(-\frac{m_i\omega_i}{\hbar(1-\mathrm{i})}x_i^2\right),$$
$$F_1(x_i) = -2\sqrt{\pi(1-\mathrm{i})}x_i\exp\left(-\frac{m_i\omega_i}{\hbar(1-\mathrm{i})}x_i^2\right),$$
$$\ldots$$

and obtain

$$|\langle x_1x_2x_3|\varphi_{000}\rangle|^2 = \left(\frac{m_1\omega_1}{\pi\hbar}\right)^{1/2}\exp\left(-\frac{m_1\omega_1}{\hbar}x_1^2\right)\left(\frac{m_2\omega_2}{\pi\hbar}\right)^{1/2}$$
$$\times \exp\left(-\frac{m_2\omega_2}{\hbar}x_2^2\right)\left(\frac{m_3\omega_3}{\pi\hbar}\right)^{1/2}\exp\left(-\frac{m_3\omega_3}{\hbar}x_3^2\right),$$
$$|\langle x_1x_2x_3|\varphi_{100}\rangle|^2 = 2\left(\frac{m_1\omega_1}{\pi\hbar}\right)^{3/2}\exp\left(-\frac{m_1\omega_1}{\hbar}x_1^2\right)\left(\frac{m_2\omega_2}{\pi\hbar}\right)^{1/2}$$
$$\times \exp\left(-\frac{m_2\omega_2}{\hbar}x_2^2\right)\left(\frac{m_3\omega_3}{\pi\hbar}\right)^{1/2}\exp\left(-\frac{m_3\omega_3}{\hbar}x_3^2\right),$$

which are consistent with results of 3D non-coupling anisotropy harmonic oscillator.

4 Discussion

Using Weyl correspondence we derive projection operator

$$|x\rangle_{\lambda,\nu\,\lambda,\nu}\langle x| = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'dp'\delta(x-\lambda x'-\nu p')\Delta(x',p'). \tag{37}$$

We can see that the Radon transforms of Wigner operator is projection operator $|x\rangle_{\lambda,\nu\,\lambda,\nu}\langle x|$, so it is necessary to introduce eigenstate $|x\rangle_{\lambda,\nu}$ of $(\lambda\hat{x}+\nu\hat{p})$. We can obtain the mean value of state vector $|\Psi\rangle$ as follows

$$|\langle\Psi|x\rangle_{\lambda,\nu}|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'dp'\delta(x-\lambda x'-\nu p')W(x',p') \tag{38}$$

where $W(x',p') = \langle\Psi|\Delta|\Psi\rangle$ is Wigner function. It is shown that the Radon transforms of Wigner function denote a distribution probability, named marginal distribution. On the other hand, from (1) and completeness relation of $|x\rangle_{\lambda,\nu}$, we obtain

$$\exp[-\mathrm{i}g(\lambda\hat{x}+\nu\hat{p})] = \int_{-\infty}^{\infty} dx|x\rangle_{\lambda,\nu\,\lambda,\nu}\langle x|e^{-\mathrm{i}gx} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dp\Delta(x,p)e^{-\mathrm{i}g(\lambda x+\nu p)}. \tag{39}$$

Considering as a Fourier transform, so its inverse transform is

$$\Delta(x,p)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dg'|g|\int_0^{\pi}d\varphi|x'\rangle_{\lambda,\nu\,\lambda,\nu}\langle x'| \times\exp\left[-\mathrm{i}g'\left(\frac{x'}{\sqrt{\lambda^2+\nu^2}}-x\cos\varphi-p\sin\varphi\right)\right] \tag{40}$$

where $g'=g(\lambda^2+\nu^2)$, $\cos\varphi=\frac{\lambda}{\sqrt{\lambda^2+\nu^2}}$, $\sin\varphi=\frac{\nu}{\sqrt{\lambda^2+\nu^2}}$.

Equation (38) shows that the Wigner function can be calculated by a marginal distribution, namely Tomography technology [18]. Substituting the Weyl ordering form of the Wigner operator [19] $\Delta(x,p)=::\delta(p-\hat{p})\delta(x-\hat{x})::$ into (37), we obtain

$$|x\rangle_{\lambda,\nu\,\lambda,\nu}\langle x|=::\delta(x-\lambda\hat{x}-\nu\hat{p})::. \tag{41}$$

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